This note develops some of the theory of linkage of Pfister forms and its relationship to properties of Pfister neighbors. We present theory extending that found in Lam [5], largely following Hoffman [2]. After a few definitions, we state our main results. In the sequel, a field is always a field of characteristic not 2.

1 Statement of Results

Recall the following definition.

Definition 1. Two \( n \)-fold Pfister forms \( \varphi_1, \varphi_2 \) are said to be linked if there exists an \( (n-1) \)-fold Pfister form \( \sigma \) and 1-fold Pfister forms \( \tau_1, \tau_2 \) such that \( \varphi_i \simeq \sigma \tau_i, \; i = 1, 2 \). A field \( F \) is said to be \( n \)-linked if every pair of \( n \)-fold Pfister forms is linked.

One theorem we shall prove is the following.

Theorem 2. If \( F \) is \( n \)-linked, then \( I_n^{n+2}F = 0 \).

A related notion is that of when forms of dimension \( 2n+1 \) are Pfister neighbors. Specifically, we make the following definition.

Definition 3. We say that a field \( F \) satisfies \( P(n) \) if every form of dimension \( 2n+1 \) is a Pfister neighbor.

Remark 4. Any isotropic form \( q \) of dimension \( 2n+1 \) is automatically a Pfister neighbor, with hyperbolic associated Pfister form. This is because we can write \( q \simeq q' \perp H \) for some \( q' \) of dimension \( 2^{n-1} \), and then \( q \perp \langle -1 \rangle q' \) is hyperbolic of dimension \( 2^{n+1} \). Thus property \( P(n) \) is only interesting when the forms we look at are anisotropic.

It is well-known (see e.g. Theorem X.4.20 in Lam [5]) that a field is 2-linked iff it satisfies \( P(2) \). In general, the Pfister neighbor property is stronger:

Proposition 5. Fix \( n \geq 2 \). If \( F \) satisfies \( P(n) \), then \( F \) is \( n \)-linked.

Another property of a field is its \( u \)-invariant. We shall be interested in this note on a variant, the Hasse number.
Definition 6. A quadratic form $\varphi$ is said to be totally indefinite (t.i.) if it is indefinite with respect to every ordering.

Definition 7. Given a field $F$, the Hasse number $\tilde{u}(F)$ is the supremum of $\dim \varphi$, where $\varphi$ ranges over all anisotropic t.i. forms.

The Hasse number is perhaps best understood through the following local-global principle.

Remark 8. Let $q$ be a quadratic form over $F$ of dimension greater than $\tilde{u}(F)$. Then $q$ is isotropic if and only if it is isotropic at all real-completions of $F$.

Perhaps our most interesting result is the following.

Theorem 9. Fix $n \geq 2$ and suppose that $F$ satisfies $P(n)$. Then $\tilde{u}(F) \leq 2^{n+1} + 2^n - 2$.

Theorems 2 and 9 are much simpler in the case of a nonreal field; see Theorem XI.6.21 in Lam [5]. As a consequence of our more general results, we will obtain the following corollaries concerning “going up” of the Pfister neighbor property.

Corollary 10. Fix $n \geq 2$. If $F$ satisfies $P(n)$, then $F$ also satisfies $P(m)$ for all $m \geq n + 2$.

Corollary 11. If $F$ is 2-linked, then $F$ satisfies $P(n)$ for all $n \geq 2$.

In closing this section, we observe that it is not known whether $P(n)$ implies $P(n + 1)$ in general. (This is conjectured by Hoffman; see Conjecture 4.9 and Corollary 4.10 in [2]. As [2] is a note from 2010, presumably the question is still unresolved.) We also observe that Corollary 11 admits a simpler proof in the case $n = 2$. We shall discuss this proof at the end of the note.

2 Linkage

For completeness, we first repeat two simple results about linkage. Both results are contained in Corollary X.6.26 in Lam [5], and both have easy proofs (although the proof of “if” in the second statement requires the Arason-Pfister Hauptsatz).

Lemma 12. Fix $n \geq 2$. If $F$ is $n$-linked, then $F$ is $m$-linked for all $m \geq n$. Also, $F$ is $n$-linked iff any form in $I^n F$ is congruent modulo $I^{n+1} F$ to an $n$-fold Pfister form.

Next we need a simple observation which appears to be taken for granted in the literature; it is implied by the theory that is proved, it is used, but it is not stated.

Lemma 13. Let $\varphi_1$ and $\varphi_2$ be $n$-fold Pfister forms, and suppose $\sigma$ is a common $r$-fold Pfister subform; that is, suppose there exist $\tau_1, \tau_2$ such that $\varphi_i \simeq \sigma \tau_i$ for $i = 1, 2$. If $\varphi_1, \varphi_2$ are linked, then they are linked “through $\sigma$”, i.e. there exists an $(n-1-r)$-fold Pfister form $\sigma'$ and 1-fold Pfister forms $\tau'_1, \tau'_2$ such that $\varphi_i \simeq \sigma \sigma' \tau'_i$ for $i = 1, 2$.  

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Proof. This is implied by Theorem X.5.13 in Lam [5], a standard result in the Elman-Lam theory of linkage. That result says that, given two \( n \)-fold Pfister forms \( \varphi, \gamma \), the Witt index of \( \varphi \perp \langle -1 \rangle \gamma \) is exactly \( 2^r \), where \( r \) is the linkage number. The proof proceeds by inductively building a linkage. If we simply begin the induction with \( \sigma \), then we obtain the stated result.

Building on this, we get the following. This is a mild strengthening of Example 3 on pg. 1178 in Elman and Lam [4].

Proposition 14. Fix \( n \geq 2 \), and suppose that \( F \) is \( n \)-linked. Let \( \varphi \) be an \((n+2)\)-fold Pfister form which is a multiple of \( \sigma \), an \( n \)-fold Pfister form. Then \( \varphi \) is also a multiple of \( \langle \langle 1 \rangle \rangle \sigma \), i.e. there exists a \( 1 \)-fold Pfister form \( \tau \) such that \( \varphi \simeq \langle \langle 1 \rangle \rangle \sigma \tau \).

Proof. Write \( \varphi \simeq \langle \langle a_1, \ldots, a_n, a_{n+1}, a_{n+2} \rangle \rangle \) where \( \sigma \simeq \langle \langle a_1, \ldots, a_n \rangle \rangle \). Linking \( \sigma \) to \( \langle \langle a_3, \ldots, a_n, a_{n+1}, a_{n+2} \rangle \rangle \) using Lemma 13, we can write \( \sigma \simeq \langle \langle x, y, a_3, \ldots, a_n \rangle \rangle \) and \( \langle \langle a_3, \ldots, a_n, x, z \rangle \rangle \simeq \langle \langle a_3, \ldots, a_n, x, z \rangle \rangle \) for some \( x, y, z \in F \). Thus

\[
\varphi \simeq \langle \langle a_1, a_2, a_3, \ldots, a_n, a_{n+1}, a_{n+2} \rangle \rangle \simeq \langle \langle x, y, a_3, \ldots, a_n, x, z \rangle \rangle.
\]

Now \( \langle \langle x, x \rangle \rangle \simeq \langle \langle 1, x \rangle \rangle \). Thus

\[
\varphi \simeq \langle \langle 1 \rangle \rangle \langle \langle x, y, a_3, \ldots, a_n \rangle \rangle \langle \langle z \rangle \rangle \simeq \langle \langle 1 \rangle \rangle \langle \langle z \rangle \rangle,
\]

as desired.

We are now in position to prove the following, which is obviously leading towards Theorem 2.

Proposition 15. Fix \( n \geq 2 \). If \( F \) is \( n \)-linked, then any \((n+2)\)-fold t.i. Pfister form is hyperbolic.

Proof. Let \( \varphi \) be such a form. By linking \( \varphi \) with \( \langle \langle 1 \rangle \rangle^{n+2} \), we obtain an expression \( \varphi \simeq \sigma \langle \langle x \rangle \rangle \) where \( \sigma \) is an \((n+1)\)-fold Pfister form (which is a subform of \( \langle \langle 1 \rangle \rangle^{n+2} \)) and \( x \in F \). Now \( \sigma \) must be (totally) positive definite, as it is a subform of \( \langle \langle 1 \rangle \rangle^{n+2} \). As \( \varphi \) is t.i., we deduce that \( x \) must be negative at all orderings. Thus \( -x \) is positive at all orderings, so it is a sum of squares.

Next, we argue by induction on \( k \) that, given any \( y \in D_F(k(1)) \), \( \varphi \) is a multiple of \( \langle \langle y, x \rangle \rangle \). The base case of \( k = 1 \) (when \( y \) is a square) is just Proposition 14. For the inductive step, suppose \( y = y' + z \) is a sum of \( k \) squares, where \( y' \) is a sum of \( k-1 \) squares and \( z \) is a square. By induction, we can write \( \varphi \) as a multiple of \( \langle \langle y', x \rangle \rangle \). Now by Proposition 14 (using \( n \geq 2 \)), we can then write \( \varphi \) as a multiple of \( \langle \langle 1, y', x \rangle \rangle \). One verifies easily that \( \langle \langle 1, y', x \rangle \rangle \) is a multiple of \( \langle \langle y \rangle \rangle \), establishing that \( \varphi \) is a multiple of \( \langle \langle y, x \rangle \rangle \). This proves the claim.

Applying the claim, we deduce that \( \varphi \) is a multiple of \( \langle \langle -x, x \rangle \rangle \). But that, of course, is hyperbolic, finishing the proof.

This “self-strengthens” to prove the full result; see for instance Corollary 2.8 on pg. 130 of Elman and Lam [3].
Proof of Theorem 2: Fix $\varphi \in I_n^\ell + 2 F$. By Lemma 12, there is an $(n+2)$-fold Pfister form $\varphi_{n+2}$ such that $\varphi \equiv \varphi_{n+2} \mod I^{n+3}$. Consider the total signature of $\varphi - \varphi_{n+2}$. At any ordering $\alpha$, $\text{sgn}_\alpha(\varphi_{n+2}) \in \{0, 2n+2\}$. But $\text{sgn}_\alpha(\varphi) = 0$ as $\varphi$ is torsion, and $\text{sgn}_\alpha(\varphi - \varphi_{n+2}) \in 2^{n+3} \mathbb{Z}$ as $\varphi - \varphi_{n+2} \in I^{n+3}$. Putting this together, we deduce that $\text{sgn}_\alpha(\varphi_{n+2}) = 0$ for all $\alpha$, and so $\varphi_{n+2}$ is torsion. Proposition 15 then implies that $\varphi_{n+2}$ is hyperbolic, which then tells us that $\varphi \in I_n^\ell + 3 F$. Repeating this argument inductively, we deduce that $\varphi \in \cap_{m \geq 0} I_m F$; but then $\varphi$ is hyperbolic by the Arason-Pfister Hauptsatz.

Incidentally, we could also deduce Theorem 2 from Proposition 15 using the fact that $I_n^\ell + 2 F$ is additively generated by the $(n+2)$-fold torsion Pfister forms. This fact, however, requires the Milnor Conjectures and is thus quite deep: see Arason and Elman [1].

To close this section, we prove an analogue of Lemma 12 for the Pfister neighbor property $P(n)$, and use it to prove Proposition 5.

Lemma 16. Fix $n \geq 2$. If $F$ satisfies $P(n)$, then given any form $\varphi$ over $F$, there exists a form $\psi$ of dimension $\leq 2n$ such that $\varphi \equiv \psi \mod I^{n+1} F$.

Proof. We prove the statement by induction on $\dim \varphi$. If $\dim \varphi \leq 2n$ then we can just take $\psi = \varphi$. Suppose instead that $\dim \varphi \geq 2n + 1$. Write $\varphi \simeq \sigma \perp \tau$ where $\dim \sigma = 2n + 1$. By $P(n)$, $\sigma$ is a Pfister neighbor: thus there exists a form $\sigma'$ of dimension $2^n$ such that $\sigma \perp \sigma'$ is similar to an $(n+1)$-fold Pfister form. In particular, $\sigma \equiv (-1) \sigma' \mod I^{n+1} F$, so $\varphi \equiv \sigma' \perp \tau$. Applying the inductive hypothesis to $\sigma' \perp \tau$, we get the desired result for $\varphi$.

Incidentally, Lemma 16 (like Lemma 12) is actually a bijection. However, the direction we proved is all we need for our purposes.

Proof of Proposition 5: Let $\varphi \in I^n F$ be arbitrary. By Lemma 16, there exists a form $\psi$ of dimension $\leq 2n$ such that $\varphi \equiv \psi \mod I^{n+1} F$. By adding in hyperbolic planes as needed, we may assume $\dim \psi = 2n$. Now $\varphi \in I^n F$ implies $\psi \in I^n F$, and the Arason-Pfister Hauptsatz tells us that the only $2^n$-dimensional forms in $I^n F$ are (multiples of) Pfister forms. Thus, replacing $\psi$ by an appropriate multiple (which does not change its residue modulo $I^{n+1} F$), we may assume $\psi$ is an $n$-fold Pfister form. We have verified the condition of Lemma 12, so $F$ is indeed $n$-linked.

3 Signature Properties

In order to prove Theorem 9, we shall need some properties relating the signature of forms to the elements they represent. Specifically, we shall want to know that, for instance, a t.i. form represents an element which is totally positive and an element which is totally negative. This is not always true, so we define the relevant properties of fields and check that fields with Pfister neighbor properties satisfy them.
Our working definitions are as follows. The properties and their names, although not the exact definitions, follow Hoffman [2] and are somewhat standard.

**Definition 17.** Let $F$ be a field. We say that $F$ satisfies SAP (strong approximation property), $S_1$, and/or ED (effective diagonalization) if the following conditions are satisfied for all binary forms $q$ over $F$:

(SAP) there exists $x \in \hat{F}$ which is positive at exactly the orderings where $q$ is positive definite;

($S_1$) if $q$ is torsion (equivalently, t.i.), then $q$ represents a totally negative element $x \in \hat{F}$; and

(ED) $q$ represents an element $x \in \hat{F}$ which is positive at exactly the orderings where $q$ is positive definite.

As stated, these definitions are not the usual ones; they are usually given in “self-strengthened” form. For instance, SAP is usually defined as the property that, given any two disjoint closed subsets $U, V$ of $X_F$, there exists $x \in \hat{F}$ such that $U \subseteq H(a)$ and $V \subseteq H(-a)$. This turns out to be equivalent to our definition above. Another equivalent definition is that SAP is the existence of a scalar which is positive exactly when a given form has a given signature; this is often how we will apply it. In addition to SAP, effective diagonalization is usually stated differently: we record the more common definition (which also makes the name more sensible) in the following lemma.

**Lemma 18.** A field $F$ satisfies ED iff for every quadratic form $q$, there exists a diagonalization $q \simeq \langle a_1, \ldots, a_n \rangle$ such that $H(a_1) \subseteq H(a_2) \subseteq \cdots \subseteq H(a_n)$.

**Proof.** The “if” direction is trivial. For “only if”, begin with any diagonalization $\langle b_1, \ldots, b_n \rangle$ of $q$. Apply the definition of ED to $\langle b_1, b_2 \rangle$, obtaining a “maximally negative” element $b'$. Then apply the definition of ED to $\langle b', b_3 \rangle$, obtaining $b''$, and so on. The resulting element $b^{(n-1)}$ is the coefficient $a_1$ in the desired diagonalization of $q$. Take its orthogonal complement and induct. 

**Corollary 19.** If $F$ satisfies ED, then every t.i. quadratic form represents a subform $\langle p, -p' \rangle$ where $p, p'$ are totally positive.

**Proof.** Let $q$ be a t.i. form of dimension $n$, and diagonalize $q$ as in Lemma 18. Set $p = a_n$ and $p' = -a_1$. 

We shall mainly be interested in ED, although it is easier to verify the two weaker conditions of SAP and $S_1$. After a lemma, we show that SAP and $S_1$ together imply ED. This argument closely follows Theorem 2.1 in Hoffman [2].

**Lemma 20.** If $F$ satisfies $S_1$, $u \in \hat{F}$ is a sum of squares, $a \in \hat{F}$, and $x \in D_F((1, au))$, then there exists a sum of squares $v \in \hat{F}$ such that $xv \in D_F((1, a))$.

**Proof.** Consider the form $q = (1, au, -a)$. As $x \in D_F((1, au))$, $q \simeq \langle x, au, -a \rangle$. But also, noting that $-x \cdot \langle au, -a \rangle$ is torsion, $S_1$ tells us that there exists a totally negative element $-u'$ (so $u'$ is a sum of squares) which it represents.
Thus \( q \simeq \langle 1, xu', -xu'u \rangle \). Comparing our expressions for \( q \), we can rearrange to find
\[
\langle 1, a \rangle = \langle x, aux, -xu', xu'u \rangle \in W(F).
\]
In particular, the form on the right side above is isotropic. Canceling \( x \), we deduce that \( \langle 1, u' \rangle \) and \( \langle u', -au \rangle \) represent a common element, say \( w \in \hat{F} \).
Now \( u \) and \( u' \) are sums of squares, so \( u'u \) is as well, and thus \( w \in D_F(\langle 1, u' \rangle) \) is also a sum of squares. Now \( \langle 1, u' \rangle \simeq \langle w, u'u'w \rangle \) and \( \langle u', -au \rangle \simeq \langle w, -au'uw \rangle \), so
\[
\langle x, aux, -xu', xu'u \rangle \simeq \langle xu, xu'uw, -xw, xau'uw \rangle \simeq \mathbb{H} \perp \langle xu'uw, xau'uw \rangle.
\]
As this equals \( \langle 1, a \rangle \) in \( W(F) \), we deduce that \( \langle 1, a \rangle \simeq xwu' = \langle 1, a \rangle \). Hence \( \langle 1, a \rangle \) represents \( xwu'u \); setting \( v = au'u \) (which is indeed a sum of squares, as it is a product of such), we get the desired result.

As an aside, Lemma 20 is actually an equivalent characterization of \( S_1 \). The direction we proved is all we need for the following.

**Proposition 21.** A field \( F \) satisfies ED iff it satisfies SAP and \( S_1 \).

**Proof.** It is clear from our definitions that SAP and \( S_1 \) are implied by ED, so we only need to show the other direction. Let \( \langle a, b \rangle \) be any binary quadratic form. By SAP, we can find \( c \in \hat{F} \) which is positive exactly when \( \langle a, b \rangle \) is positive definite; thus \( \langle a, b, -c \rangle \) is totally indefinite. Set \( q = \langle a, b, -c, -abc \rangle \simeq \langle -ac, -bc \rangle \). This is indefinite everywhere, so (being similar to a Pfister form) it has zero signature everywhere, and thus it is 2-primary torsion. Choose \( n \geq 0 \) such that \( 2^n q \) is hyperbolic. Thus the subform (also, Pfister neighbor) \( 2^n \langle a, b \rangle \perp \langle -c \rangle \) of more than half the dimension must be isotropic. In particular, \( 2^n \langle a, b \rangle \) represents \( c \). It follows that there are \( u, v \in D_F(2^n(1)) \) such that \( \langle au, vb \rangle \) represents \( c \). Thus \( \langle 1, uvab \rangle \) represents \( uvac \). Now using \( S_1 \) (through Lemma 20), we deduce that there is a sum of squares \( w \in \hat{F} \) such that \( \langle 1, ab \rangle \) represents \( uvac \). As \( uuv \) is a sum of squares, it is totally positive, and so \( H(uvac) = H(c) \). (Hence \( uvac \) is positive at exactly the “right” orderings.) As \( \langle a, b \rangle \) represents \( uvac \), we have verified ED.

Our goal for the remainder of the section is to show that fields satisfying \( \mathcal{P}(n) \) have ED. By Proposition 21, we can prove this in two steps.

**Lemma 22.** Fix \( n \geq 2 \). If \( F \) is \( n \)-linked, then \( F \) satisfies SAP.

**Proof.** Let \( \langle a, b \rangle \) be a binary quadratic form. This is positive definite exactly when \( a \) and \( b \) are both positive, which is equivalent to \( \langle 1 \rangle^{n-2} \langle a, b \rangle \) being positive definite. By linkage with \( \langle 1 \rangle^n \), we can write \( \langle 1 \rangle^{n-2} \langle a, b \rangle \simeq \sigma \langle c \rangle \) where \( \sigma \) is a subform of \( \langle 1 \rangle^n \) (so is positive definite) and \( c \in \hat{F} \). Hence \( \langle 1 \rangle^{n-2} \langle a, b \rangle \) is positive definite exactly when \( c \) is positive. Unwinding all of this, \( c \) is positive exactly when \( \langle a, b \rangle \) is positive definite, proving SAP.

**Lemma 23.** Fix \( n \geq 2 \). If \( F \) satisfies \( \mathcal{P}(n) \), then it satisfies \( S_1 \).
Proof. Let \( \langle a, b \rangle \) be any torsion binary form. Consider the \((2^n + 1)\)-dimensional form \( q = (2^n - 1)(1) \perp \langle a, b \rangle \). This is a Pfister neighbor by \( \mathcal{P}(n) \), say \( q \subseteq c \cdot \pi \) where \( \pi \) is an \((n + 1)\)-fold Pfister form. As \( q \) represents 1, \( c \cdot q \subseteq \pi \) represents \( c \). As \( \pi \) is Pfister, the represented element \( c \) is a similarity factor; thus \( q \subseteq \pi \).

If \( \pi \) is isotropic, then it is hyperbolic and so the neighbor \( q \) is isotropic; this implies that \( \langle a, b \rangle \) and \((2^n - 1)(-1)\) represent a common element. In other words, \( \langle a, b \rangle \) represents a totally negative element, which proves \( S_1 \). Suppose this does not happen, i.e. suppose \( \pi \) is anistropic. Pass to the function field \( F[2^n(1)] \) for the Pfister form \( 2^n(1) \). Over this field, \( 2^n(1) \) becomes isotropic, so hyperbolic, so the Pfister neighbor \((2^n - 1)(1)\) becomes isotropic, so \( q \) and thus \( \pi \) becomes isotropic. By Corollary X.4.13 in Lam [5], this implies that \( \pi \) contains \( 2^n(1) \) as a subform.

Now \( \pi \) contains \( \langle a, b \rangle \), which is totally indefinite. Thus \( \pi \) is t.i., so \( \pi \) is an Pfister \( \pi \) (being Pfister) it has zero total signature and thus is torsion. Hence \( \langle 1 \rangle \pi \subseteq I^{n+2}F \) is hyperbolic by (Proposition 5 plus) Theorem 2. Now \( \langle 1 \rangle \pi \simeq \pi \perp \pi \) contains the subform

\[
2^n(1) \perp ((2^n - 1)(1) \perp \langle a, b \rangle) \simeq (2^n+1 - 1)(1) \perp \langle a, b \rangle
\]

of dimension \( 2^{n+1} + 1 \), which must be isotropic. As above, this isotropy implies that \( \langle a, b \rangle \) represents a totally negative element, as desired. \( \square \)

**Corollary 24.** If \( F \) satisfies \( \mathcal{P}(n) \) for some \( n \geq 2 \), then \( F \) has ED.

**Proof.** By Proposition 5 plus Lemma 22, \( F \) satisfies SAP. By Lemma 23, \( F \) satisfies \( S_1 \). Thus it has ED by Proposition 21. \( \square \)

## 4 The Hasse Number

Our goal in this section is to prove Theorem 9 and Corollary 10.

**Proof of Theorem 9:** Suppose for contradiction that \( \tilde{u}(F) > 2^{n+1} + 2^n - 2 \), and let \( q \) be an anistotropic t.i. form of dimension larger than this bound. By Corollaries 19 and 24, \( q \) contains a binary torsion subform. By passing to a subform of \( q \) containing this fixed binary form, we may assume without loss of generality that the anistotropic, t.i. form \( q \) has \( \dim q = 2^{n+1} + 2^n - 1 \).

By Lemma 16, there exists a form \( \psi \) of dimension \( \leq 2^n \) such that \( q \equiv \psi \mod I^{n+2}F \). As \( \dim q \) is odd, so must be \( \dim \psi \), so actually \( \dim \psi < 2^n \). Define \( q' = q \perp (-1)\psi \). As \( q' \) is a form of dimension less than \( 2^{n+2} \) in \( I^{n+1}F \), its signature at any ordering is either 0 or \( \pm 2^{n+1} \). By \((n + 1)\)-linkage, there exists a \((n + 1)\)-fold Pfister form \( \varphi \) such that \( \varphi \equiv q' \mod I^{n+2}F \). Being a Pfister form, \( \varphi \) has signature either 0 or \( 2^{n+1} \) (again, at any ordering). Now \( \varphi \equiv q' \mod I^{n+2}F \) implies \( \text{sgn}(\varphi) \equiv \text{sgn}(q') \mod 2^{n+2} \). We conclude from all this that \( |\text{sgn}(\varphi)| = |\text{sgn}(q')| \). By Lemma 22, there exists \( a \in \tilde{F} \) such that \( a \) is positive at exactly the orderings such that \( \text{sgn}(q') > 0 \). Hence \( a \cdot \varphi \) and \( q' \) have the same total signature. They are also equivalent modulo \( I^{n+2}F \),
so their difference is an element of $I_{n+2}^+(F)$. By Theorem 2, we conclude that $q' = a \cdot \varphi \in W(F)$.

Let $\pi$ be the anisotropic part of $q'$. The equality $q' = a \cdot \varphi \in W(F)$ tells us that $\dim \pi \leq \dim \varphi = 2^{n+1}$. But because $q$ is anisotropic, $\dim \pi \geq \dim q - \dim \psi \geq 2^{n+1}$, with equality iff $\psi$ is a subform of $q$ (and $\dim \psi = 2^n - 1$). Equality must hold, so $\pi \cong \psi \perp \pi$, $\dim \pi = 2^{n+1}$, and thus (because $\pi = q' = a \cdot \varphi \in W(F)$) we have $\pi \cong a \cdot \varphi$.

Now $q$ is t.i., so $a \cdot q \cong (a \cdot \psi) \perp \varphi$ is also totally indefinite. Recall from above that the Pfister form $\varphi$ is either positive definite or has zero signature. As $F$ has ED (Corollary 24), Lemma 18 tells us that $a \cdot \psi$ represents an element $b \in \hat{F}$ which is negative whenever $\varphi$ is positive definite. Then $a \cdot q$ contains $(b) \perp \varphi$, a $(2^{n+1} + 1)$-dimensional t.i. special Pfister neighbor. Its associated Pfister form is in $I_{n+2}^+(F)$, so it is hyperbolic by Theorem 2. Thus the Pfister neighbor $(b) \perp \varphi$ is isotropic, so $a \cdot q$ (and thus $q$) is also isotropic. This provides a contradiction and completes the proof.

Our remaining task for this section is to prove Corollary 10. We do this by giving a general sufficient condition for $P(n)$ in terms of the Hasse number. The first step for this is verifying that fields with finite Hasse number satisfy SAP. This is not really necessary for proving Corollary 10, as the fields we are interested in satisfy SAP by Lemma 22. However, it is not hard to prove directly that even ED holds in this setting, so we do so now.

Lemma 25. Suppose $\bar{u}(F) < \infty$. Then $F$ has ED; in particular, it satisfies SAP and $S_1$.

Proof. It suffices to check SAP and $S_1$ by Proposition 21. We first verify SAP. Let $(a, b)$ be a binary form. Now $\langle a, b \rangle = (1, a, b, ab)$ is a Pfister form, so has signature either 0 or 4. The latter case occurs when $a$ and $b$ are both positive, i.e. when $(a, b)$ is positive definite. Thus $\langle a, b, ab \rangle$ has signature either $-1$ or 2, and in particular it always represents a positive element. It follows that $\langle a, b, ab \rangle \perp n(-1)$ is t.i. for any $n \geq 1$. Choosing $n$ such that $n + 3 > \bar{u}(F)$, the resulting form must be isotropic. This tells us that $\langle a, b, ab \rangle$ and $n(1)$ represent a common element $c \in \hat{F}$, which of course must be totally positive. Write $\langle a, b, ab \rangle \simeq \langle c \rangle \perp q$. Then the signature of $q$ is either $-2$ or 2, i.e. $q$ is always definite, and it is positive definite exactly when $(a, b)$ is positive definite. Hence any element represented by $q$ has the right signature properties, proving SAP.

We conclude by proving $S_1$, which is quite easy. Let $q$ be any torsion binary form. Then $q \perp n(1)$ is t.i. for any $n \geq 0$. Taking $n$ large enough, this t.i. form must be isotropic. Hence $q$ represents a totally negative element, as desired.

The following proposition gives the sufficient condition for $P(n)$.

Proposition 26. Fix a positive integer $n$ and suppose that $\bar{u}(F) \leq 2^n$. Then $F$ satisfies $P(n)$.

Proof. Let $q$ be a form of dimension $2^n + 1$ over $F$. As $F$ satisfies SAP by Lemma 25, we can find a scalar which is negative exactly when $q$ has negative signature.
Multiplying $q$ by this scalar, we may assume without loss of generality that $q$ has nonnegative signature everywhere. By Remark 4, it suffices to consider the case when $q$ is anisotropic. As $\dim q > \tilde{u}(F)$, $q$ must not be t.i., so there is at least one ordering at which $q$ is positive definite.

Again applying SAP, there exists a scalar $c \in \bar{F}$ which is positive at exactly the orderings where $q$ is positive definite. Define $\varphi = \langle 1 \rangle^n \langle c \rangle$ and $\pi = \varphi \perp (-1)q$. Now $\text{sgn}(\varphi)$ equals $2^{n+1}$ when $\text{sgn}(q) = 2^n + 1$, and otherwise $\text{sgn}(\varphi) = 0$. In the former case, $\text{sgn}(\pi) = 2^n - 1$, and in the latter case $\text{sgn}(\pi) = -\text{sgn}(q) \geq -(2^n - 1)$. Either way, we deduce that the anisotropic part of $\pi$ at any real-closure of $F$ has dimension at most $2^n - 1$. Repeatedly applying Remark 8, this tells us that the anisotropic part of $\pi$ (over $F$) also has dimension at most $2^n - 1$. But this bound equals $\dim \varphi - \dim q$, and $\varphi$ is anistropic (as it is positive definite at some ordering). We conclude that $\dim(\pi)_{an} = 2^n - 1$ and so $q \subset \varphi$. Hence $q$ is a Pfister neighbor, as desired.

**Proof of Corollary 10:** By Theorem 9, $\tilde{u}(F) \leq 2^{n+1} + 2^n - 2 \leq 2^{n+2} \leq 2^n$ for any $m \geq n + 2$. Thus $F$ satisfies $\mathcal{P}(m)$ for any such $m$ by Proposition 26. □

## 5 Special Case of Linked Fields

In this section we prove the one special case needed to finish Corollary 11, and then give a simpler proof of the general case. More specifically, having already established Corollary 11, all we need to complete Corollary 11 is to check that 2-linked fields satisfy $\mathcal{P}(3)$. We do this by a fairly direct argument. Recall that the $\mathcal{P}(2)$ property, which 2-linked fields satisfy, self-amplifies to the following: every 5-dimensional form is a special Pfister neighbor. (This is part of Theorem X.4.20 in Lam [5].) The proof is now fairly straightforward.

**Lemma 27.** If $F$ is 2-linked, then $F$ satisfies $\mathcal{P}(3)$.

**Proof.** Let $q$ be any 9-dimensional $F$-form, and let $d = d(q)$ be its determinant. Fix some decomposition $q \simeq q_4 \perp q_5$, where $q_4$ and $q_5$ are 4-dimensional and 5-dimensional forms, respectively. Applying the special Pfister neighbor property to $q_5$, there exist $a, b, c \in \bar{F}$ such that $q_5 \simeq a \cdot \langle b, c \rangle \perp \langle d(q_5) \rangle$. Applying the same argument to $q_4 \perp \langle d(q_5) \rangle$, we find $e, f, g \in \bar{F}$ such that $q_4 \perp \langle d(q_5) \rangle \simeq e \cdot \langle f, g \rangle \perp \langle d(q_4)d(q_5) \rangle$. Of course $d(q_4)d(q_5) = d$. By the 2-linked property, we may assume without loss of generality that $b = f$. Putting this all together,

$$q \subset q \perp \langle bd \rangle \simeq \langle b \rangle \langle a \cdot \langle c \rangle \perp e \cdot \langle g \rangle \perp \langle d \rangle \rangle.$$  

The 5-dimensional form $a \cdot \langle c \rangle \perp e \cdot \langle g \rangle \perp \langle d \rangle$ is contained in a (multiple of a) 3-fold Pfister form by $\mathcal{P}(2)$, and then $q$ is contained in $\langle b \rangle$ times that. □

This argument relied in a crucial way on the special Pfister neighbor property, which is not always satisfied for larger $n$. Thus it does not seem to generalize to an argument proving $\mathcal{P}(n) \implies \mathcal{P}(n+1)$. This leaves open the possibility
that it can be adapted to prove $P(2) \implies P(n)$ (for all $n$), but we have not seen how to do this.

It is, however, possible to streamline the argument given in the previous sections for $P(2) \implies P(n)$ for $n \geq 4$. This gives us a proof of Corollary 11 that relies only (in terms of theory developed in this note) on Theorem 2 (actually, just Proposition 15), Lemma 22, and Lemma 27. It thus avoids much of the theory developed here. (However, it is still a nontrivial argument! Given that Corollary 11 was posed as an exercise in Lam [5], one might reasonably think that a much simpler proof exists.)

Proof of Corollary 11: With Lemma 27 in hand, we may restrict our attention to proving $P(n)$ for $n \geq 4$. So fix $n \geq 4$. Let $q$ be a form of dimension $2^n + 1$. We shall define a sequence of forms

$$q = \psi_0 \subset \psi_1 \subset \cdots \subset \psi_n \subset \psi_{n+1}$$

where, for each $i$, the form $\psi_i$ has dimension $(2^n+1)+(1+2+\cdots+2^{i-1}) = 2^n+2^i$ and $\psi_i \in I^n F$. We do this inductively as follows. For each $i = 0, 1, \ldots, n$, by Lemma 12 there exists an $i$-fold Pfister form $\varphi_i$ such that $\psi_i \equiv \varphi_i \mod I^{n+1} F$. (Technically we are only applying Lemma 12 for $i \geq 2$, but for $i = 0, 1$ the claim is obvious.) We define $\psi_{i+1} = \psi_i \perp (a_i)\varphi_i$ for an appropriately chosen $a_i \in \hat{F}$. Note that, no matter what scalars $a_i$ we choose, the asserted properties about the forms $\psi_i$ are satisfied.

We choose the scalars $a_i$ by signature considerations. Specifically, the rules are as follows. Fix for the moment some ordering of $F$. If $\varphi_i$ is definite, then we shall take $a_i$ to have the same sign (i.e. if $\varphi_i$ is positive definite, then we want $a_i$ to be positive). Otherwise, we take $a_i$ to have the opposite sign from the signature of $\varphi_i$. Note that there exists some $a_i \in \hat{F}$ with the desired signs at all the orderings by Lemma 22, so this is well-defined.

It remains to examine what we have. This follows the concepts in the proof of Theorem 9. Let $\alpha$ be an ordering of $F$. For any $i$, $\text{sgn}_\alpha(\varphi_i) \in \{0, 2^i\}$ as $\varphi_i$ is $i$-fold Pfister. Moreover, we must have $\text{sgn}_\alpha(\psi_i) \equiv \text{sgn}_\alpha(\varphi_i) \mod 2^{n+1}$ as $\psi_i \equiv \varphi_i \mod I^{n+1} F$. Thus $\text{sgn}_\alpha(\psi_i)$ is actually uniquely determined by $\text{sgn}_\alpha(\psi_i)$.

Thanks to this observation, and our choice of the signs of $a_i$, we have the following situation. If $q$ is $\alpha$-definite, i.e. $|\text{sgn}_\alpha(q)| = 2^n + 1$, then $\text{sgn}_\alpha(\psi_i) = 2^n + 2^i$ for all $i = 0, 1, \ldots, n$. In particular, $\text{sgn}_\alpha(\psi_n) = 2^{n+1}$. Otherwise, if $q$ is not $\alpha$-definite, then $|\text{sgn}_\alpha(q)| \leq 2^n - 1$. One verifies that $|\text{sgn}_\alpha(\psi_i)|$ is decreasing in $i$, and in particular $\text{sgn}_\alpha(\psi_n) = 0$.

The above arguments show that $2^{n+1}$ divides the signature of $\psi_n$ at all orderings, which in turn implies that the $n$-fold Pfister form $\varphi_n$ has zero total signature. But by Proposition 15, every torsion $n$-fold Pfister form is hyperbolic. Thus, working in the Witt ring, $\psi_n = \psi_{n+1} \in I^{n+1} F$. Being a $2^{n+1}$-dimensional form in $I^{n+1} F$, $\psi_n$ is a multiple of a Pfister form by the Arason-Pfister Hauptsatz. As this (multiple of a) Pfister form contains $q$, $q$ is indeed a Pfister neighbor. □
We conclude with some remarks concerning this proof. It uses in a crucial way the conclusion of Lemma 12 for all $i$, which is only true if $F$ is 2-linked. Thus we suspect that the proof does not yield a direct proof of the stronger Corollary 10. Moreover, the proof does not seem to handle property $\mathcal{P}(3)$, as linked fields may have anisotropic torsion 3-fold Pfister forms. Thus, while this is more direct than the proof we gave of Corollary 10, it does not seem to generalize in any fruitful way.

References


